

# ON THE VECTOR BUNDLES ASSOCIATED TO THE IRREDUCIBLE REPRESENTATIONS OF COCOMPACT LATTICES OF $\mathrm{SL}(2, \mathbb{C})$

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**ABSTRACT.** In this continuation of [BM], we prove the following: Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$  be a cocompact lattice, and let  $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$  be an irreducible representation. Then the holomorphic vector bundle  $E_\rho \longrightarrow \mathrm{SL}(2, \mathbb{C})/\Gamma$  associated to  $\rho$  is polystable. The compact complex manifold  $\mathrm{SL}(2, \mathbb{C})/\Gamma$  has natural Hermitian structures; the polystability of  $E_\rho$  is with respect to these natural Hermitian structures. In [BM] it was shown that if  $\rho(\Gamma) \subset \mathrm{U}(r)$ , then  $E_\rho$  equipped with the Hermitian structure given by  $\rho$ , and  $\mathrm{SL}(2, \mathbb{C})/\Gamma$  equipped with a natural Hermitian structure, together produce a solution of the Strominger system of equations. A polystable vector bundle also has a natural Hermitian structure, which is known as the Hermitian–Yang–Mills structure. It would be interesting to find similar applications of the Hermitian–Yang–Mills structure on the above polystable vector bundle  $E_\rho$ . We show that the polystable vector bundle  $E_\rho$  is not stable in general.

## 1. INTRODUCTION

We first recall the set-up, and some results, of [BM]. Let

$$\Gamma \subset \mathrm{SL}(2, \mathbb{C})$$

be a discrete cocompact subgroup. Fixing a  $\mathrm{SU}(2)$ -invariant Hermitian form on the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ , we get a Hermitian structure  $h$  on the compact complex manifold  $M := \mathrm{SL}(2, \mathbb{C})/\Gamma$ . The  $(1, 1)$ -form  $\omega_h$  on  $M$  associated to  $h$  satisfies the identity  $d\omega_h^2 = 0$ . Take any homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

This  $\rho$  produces a holomorphic vector bundle  $E_\rho$  of rank  $r$  on  $M$  equipped with a flat holomorphic connection  $\nabla^\rho$ . The homomorphism  $\rho$  is called irreducible if  $\rho(\Gamma)$  is not contained in some proper parabolic subgroup of  $\mathrm{GL}(r, \mathbb{C})$ .

If  $\rho(\Gamma) \subset \mathrm{U}(r)$ , then  $E_\rho$  is equipped with a Hermitian structure  $H^\rho$  such that the associated Chern connection is  $\nabla^\rho$ .

If

- $\rho(\Gamma) \subset \mathrm{U}(r)$  and
- $\rho(\Gamma)$  is irreducible,

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then the quadruple  $(M, h, E_\rho, H^\rho)$  satisfies the Strominger system of equations [BM, Theorem 4.6]. In particular, the vector bundle  $E_\rho$  is stable [BM, Proposition 4.5].

Now assume that  $\rho$  is irreducible, but do *not* assume that  $\rho(\Gamma) \subset \mathrm{U}(r)$ . Our aim here is to prove the following (see Theorem 2.2):

*The holomorphic vector bundle  $E_\rho$  is polystable with respect to the Hermitian structure  $h$  on  $M$ .*

It is known that under some minor condition, the group  $\Gamma$  admits some free groups of more than one generators as quotients [La, p. 3393, Theorem 2.1]. Therefore, there are many examples of pairs  $(\Gamma, \rho)$  of the above type satisfying the irreducibility condition.

Since  $E_\rho$  is polystable, the holomorphic vector bundle  $E_\rho$  has an Hermitian–Yang–Mills structure  $\mathcal{H}^\rho$  [LY] (see also [Bu]). It may be worthwhile to investigate this Hermitian structure  $\mathcal{H}^\rho$ . We should clarify that  $\mathcal{H}^\rho$  need not be flat. An Hermitian–Yang–Mills structure on a polystable vector bundle with vanishing Chern classes over a compact Kähler manifold is flat, but  $M$  is not Kähler.

It is natural to ask whether the polystable vector bundle  $E_\rho$  is stable. If we take  $\rho$  to be the inclusion of  $\Gamma$  in  $\mathrm{SL}(2, \mathbb{C})$ , then  $\rho$  is irreducible, but the associated holomorphic vector bundle  $E_\rho$  is holomorphically trivial, in particular,  $E_\rho$  is not stable (see Lemma 2.3 for the details).

Infinitesimal deformations of the complex structure of  $M$  are investigated in [Ra].

## 2. POLYSTABILITY OF ASSOCIATED VECTOR BUNDLE

The Lie algebra of  $\mathrm{SL}(2, \mathbb{C})$ , which will be denoted by  $\mathfrak{sl}(2, \mathbb{C})$ , is the space of complex  $2 \times 2$  matrices of trace zero. Consider the adjoint action of  $\mathrm{SU}(2)$  on  $\mathfrak{sl}(2, \mathbb{C})$ . Fix an inner product  $h_0$  on  $\mathfrak{sl}(2, \mathbb{C})$  preserved by this action; for example, we may take the Hermitian form  $(A, B) \mapsto \mathrm{trace}(AB^*)$  on  $\mathfrak{sl}(2, \mathbb{C})$ . Let  $h_1$  be the Hermitian structure on  $\mathrm{SL}(2, \mathbb{C})$  obtained by right–translating the Hermitian form  $h_0$  on  $T_{\mathrm{Id}}\mathrm{SL}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})$ .

Let  $\Gamma$  be a cocompact lattice in  $\mathrm{SL}(2, \mathbb{C})$ . So  $\Gamma$  is a discrete subgroup of  $\mathrm{SL}(2, \mathbb{C})$  such that the quotient

$$(2.1) \quad M := \mathrm{SL}(2, \mathbb{C})/\Gamma$$

is compact. This  $M$  is a compact complex manifold of complex dimension three. The left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on itself descends to an action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$ . We will call this action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$  the *left–translation action*. The Hermitian structure  $h_1$  on  $\mathrm{SL}(2, \mathbb{C})$  descends to an Hermitian structure on  $M$ . This descended Hermitian structure on  $M$  will be denoted by  $h$ . Let  $\omega_h$  be the  $C^\infty$   $(1, 1)$ –form on  $M$  associated to  $h$ . Then

$$d\omega_h^2 = 0$$

[BM, Corollary 4.1].

For a torsionfree nonzero coherent analytic sheaf  $F$  on  $M$ , define

$$\mathrm{degree}(F) := \int_M c_1(F) \wedge \omega_h^2 \in \mathbb{R} \quad \text{and} \quad \mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \in \mathbb{R}.$$

A torsionfree nonzero coherent analytic sheaf  $F$  on  $M$  is called *stable* (respectively, *semistable*) if for every coherent analytic subsheaf

$$V \subset F$$

such the  $\mathrm{rank}(V) \in [1, \mathrm{rank}(F) - 1]$  and the quotient  $F/V$  is torsionfree, the inequality

$$\mu(V) < \mu(F) \quad (\text{respectively, } \mu(V) \leq \mu(F))$$

holds (see [Ko, Ch. V, § 7]). A torsionfree nonzero coherent analytic sheaf  $F$  on  $M$  is called *polystable* if it is semistable and is isomorphic to a direct sum of stable sheaves.

**Remark 2.1.** Since a polystable coherent analytic sheaf  $F$  is semistable, if  $F = \bigoplus_{i=1}^{\ell} F_i$ , then  $\mu(F_i) = \mu(F)$  for all  $i$ .

Take any homomorphism

$$(2.2) \quad \rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

Let  $(E_\rho, \nabla^\rho)$  be the flat holomorphic vector bundle of rank  $r$  over  $M$  associated to the homomorphism  $\rho$ . We recall that the total space of  $E_\rho$  is the quotient of  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$  where two points

$$(z_1, v_1), (z_2, v_2) \in \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$$

are identified if there is an element  $\gamma \in \Gamma$  such that  $z_2 = z_1\gamma$  and  $v_2 = \rho(\gamma^{-1})(v_1)$ . The trivial connection on the trivial vector bundle  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r \longrightarrow \mathrm{SL}(2, \mathbb{C})$  of rank  $r$  descends to the connection  $\nabla^\rho$ . The left-translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathrm{SL}(2, \mathbb{C})$  and the trivial action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathbb{C}^r$  together define an action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ . This action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$  descends to an action

$$(2.3) \quad \tau : \mathrm{SL}(2, \mathbb{C}) \times E_\rho \longrightarrow E_\rho$$

of  $\mathrm{SL}(2, \mathbb{C})$  on the vector bundle  $E_\rho$ . The action  $\tau$  in (2.3) is clearly a lift of the left-translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$ .

The homomorphism  $\rho$  in (2.2) is called *reducible* if there a nonzero linear subspace  $S \subsetneq \mathbb{C}^r$  such that  $\rho(\Gamma)(S) = S$ . The homomorphism  $\rho$  is called *irreducible* if it is not reducible.

**Theorem 2.2.** *Assume that the homomorphism  $\rho$  in (2.2) is irreducible. Then the corresponding holomorphic vector bundle  $E_\rho$  is polystable.*

*Proof.* Since  $E_\rho$  has a flat connection, the Chern class  $c_1(\det E_\rho) = c_1(E_\rho) \in H^2(M, \mathbb{R})$  vanishes. Hence we have  $\mathrm{degree}(E_\rho) = 0$  (see [BM, Lemma 4.2]).

We will first show that  $E_\rho$  is semistable. Assume that  $E_\rho$  is not semistable. Let

$$(2.4) \quad 0 \subset W_1 \subset \cdots \subset W_{\ell-1} \subset W_\ell = E_\rho$$

be the Harder–Narasimhan filtration  $E_\rho$ ; see [Br] for the construction of the Harder–Narasimhan filtration of vector bundles on compact complex manifolds. Since  $E_\rho$  is not semistable, we have  $\ell \geq 2$  and  $W_1 \neq 0$ .

Consider the action  $\tau$  of  $\mathrm{SL}(2, \mathbb{C})$  on  $E_\rho$  constructed in (2.3). From the uniqueness of the Harder–Narasimhan filtration it follows immediately that  $\tau(\{g\} \times W_1) = W_1$  for every  $g \in \mathrm{SL}(2, \mathbb{C})$ . Therefore, we have

$$(2.5) \quad \tau(\mathrm{SL}(2, \mathbb{C}) \times W_1) = W_1.$$

Let  $C(W_1) \subsetneq M$  be the closed subset over which  $W_1$  fails to be locally free. Since  $\tau$  is a lift of the left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$ , from (2.5) we conclude that  $C(W_1)$  is preserved by the left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$ . As the left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$  is transitive, it follows that  $C(W_1)$  is the empty set. Therefore,  $W_1$  is a holomorphic vector bundle on  $M$ . Similarly, the closed proper subset of  $M$  over which  $W_1$  fails to be a subbundle of  $E_\rho$  is preserved the left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on  $M$ . Hence this subset is empty, and  $W_1$  is a holomorphic subbundle of  $E_\rho$ .

We will show that the flat connection  $\nabla^\rho$  on  $E_\rho$  preserves the subbundle  $W_1$  in (2.4).

To show that  $\nabla^\rho$  preserves  $W_1$ , first note that the flat sections of the trivial connection on the trivial vector bundle  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r \rightarrow \mathrm{SL}(2, \mathbb{C})$  are of the form

$$\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r, \quad g \mapsto (g, v_0),$$

where  $v_0 \in \mathbb{C}^r$  is independent of  $g$ . On the other hand, the image of such a section is an orbit for the action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ ; recall that the action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r$  is the diagonal one for the left–translation action of  $\mathrm{SL}(2, \mathbb{C})$  on itself and the trivial action of  $\mathrm{SL}(2, \mathbb{C})$  on  $\mathbb{C}^r$  (see the construction of  $\tau$  in (2.3)). Also, recall that the connection  $\nabla^\rho$  on  $E_\rho$  is the descent of the trivial connection on the trivial vector bundle  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^r \rightarrow \mathrm{SL}(2, \mathbb{C})$ . Combining these, from (2.5) we conclude that  $\nabla^\rho$  preserves  $W_1$ .

The homomorphism  $\rho$  is given to be irreducible. Therefore, the only holomorphic subbundles of  $E_\rho$  that are preserved by the associated connection  $\nabla^\rho$  are 0 and  $E_\rho$  itself. But  $\ell \geq 2$  and  $W_1 \neq 0$  in (2.4). So  $W_1$  neither 0 nor  $E_\rho$ .

In view of the above contradiction, we conclude that the holomorphic vector bundle  $E_\rho$  is semistable.

We will now prove that  $E_\rho$  is polystable.

Consider all nonzero coherent analytic subsheaves  $V$  of  $E_\rho$  such that

- $V$  is polystable, and
- $\mathrm{degree}(V) = 0$ .

Let

$$(2.6) \quad \mathcal{F} \subset E_\rho$$

be the coherent analytic subsheaf generated by all  $V$  satisfying the above two conditions. It is known that  $\mathcal{F}$  is polystable with  $\mu(\mathcal{F}) = \mu(E_\rho) = 0$  (see [HL, page 23, Lemma 1.5.5]). Therefore, the subsheaf  $\mathcal{F}$  is uniquely characterized as follows: the subsheaf  $\mathcal{F}$  is the unique maximal coherent analytic subsheaf of  $E_\rho$  such that

- $\mathcal{F}$  is polystable, and
- $\mathrm{degree}(\mathcal{F}) = 0$ .

Note that the quotient  $E_\rho/\mathcal{F}$  is torsionfree, because if  $T \subset E_\rho/\mathcal{F}$  is the torsion part, then  $\varphi^{-1}(T) \subset E_\rho$ , where

$$\varphi : E_\rho \longrightarrow E_\rho/\mathcal{F}$$

is the quotient map, also satisfies the above two conditions, while  $\mathcal{F} \subsetneq \varphi^{-1}(T)$  if  $T \neq 0$ .

Consider the action  $\tau$  of  $\mathrm{SL}(2, \mathbb{C})$  on  $E_\rho$  constructed in (2.3). From the above characterization of the subsheaf  $\mathcal{F}$  in (2.6) it follows immediately that

$$(2.7) \quad \tau(\mathrm{SL}(2, \mathbb{C}) \times \mathcal{F}) = \mathcal{F}.$$

As it was done for  $W_1$ , from (2.7) we conclude that  $\mathcal{F}$  is a holomorphic subbundle of  $E_\rho$ .

As it was done for  $W_1$ , from (2.7) it follows that the flat connection  $\nabla^\rho$  on  $E_\rho$  preserves the subbundle  $\mathcal{F}$  in (2.6). Since  $\rho$  is irreducible, either  $\mathcal{F} = 0$  or  $\mathcal{F} = E_\rho$ . The rank of  $\mathcal{F}$  is at least one because the semistable vector bundle  $E_\rho$  of degree zero has a nonzero stable subsheaf of degree zero. Therefore, we conclude that  $\mathcal{F} = E_\rho$ . Consequently,  $E_\rho$  is polystable.  $\square$

We may now ask whether the polystable vector bundle  $E_\rho$  in Theorem 2.2 is stable. The following lemma shows that  $E_\rho$  is not stable in general.

Let

$$(2.8) \quad \delta : \Gamma \hookrightarrow \mathrm{SL}(2, \mathbb{C})$$

be the inclusion map. This homomorphism  $\delta$  is clearly irreducible. Let  $(E_\delta, \nabla^\delta)$  be the corresponding flat holomorphic vector bundle on  $M$ .

**Lemma 2.3.** *The above holomorphic vector bundle  $E_\delta$  is holomorphically trivial.*

*Proof.* Recall that the vector bundle  $E_\delta$  is a quotient of  $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2$ . Consider the holomorphic map

$$\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \longrightarrow \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2$$

defined by  $(g, v) \mapsto (g, g(v))$ . This map descends to a holomorphic isomorphism of vector bundles

$$E_\delta \longrightarrow M \times \mathbb{C}^2$$

over  $M$ . Therefore, this descended homomorphism provides a holomorphic trivialization of  $E_\delta$ .  $\square$

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